On transfer inequalities in Diophantine approximation, II

Yann Bugeaud & Michel Laurent

1. Introduction

Let n be a positive integer and let $\Theta = (\theta_1, \ldots, \theta_n)$ be a point in $\mathbf{R}^n$. We shall assume in all the forthcoming statements that the real numbers $1, \theta_1, \ldots, \theta_n$ are linearly independent over the field $\mathbf{Q}$ of rational numbers. Khintchine's transference principle relates the sharpness of the rational simultaneous approximation to $\theta_1, \ldots, \theta_n$ with the measure of linear independence over $\mathbf{Q}$ of $1, \theta_1, \ldots, \theta_n$. Let us first quantify these notions by introducing the exponents $\omega_0(\Theta)$ and $\omega_{n-1}(\Theta)$ (the meaning of the indices 0 and $n-1$ will be explained afterwards).

Definition 1. *We denote respectively by $\omega_0(\Theta)$ and $\omega_{n-1}(\Theta)$ the supremum, possibly infinite, of the real numbers ω for which there exist infinitely many integer $(n+1)$-tuples $(x_0, \ldots, x_n)$ satisfying respectively the inequation*

$$\max_{1\le i\le n} |x_0\theta_i - x_i| \le \Big(\max_{0\le i\le n} |x_i|\Big)^{-\omega} \quad \textit{or} \quad |x_0 + x_1\theta_1 + \cdots + x_n\theta_n| \le \Big(\max_{0\le i\le n} |x_i|\Big)^{-\omega}.$$

Now we can state Khintchine's transference principle [12] (see [15] for an alternative proof, and the monographs [6, 18, 8]) as follows:

Theorem K. *The inequalities*

$$\frac{\omega_{n-1}(\Theta)}{(n-1)\omega_{n-1}(\Theta)+n} \le \omega_0(\Theta) \le \frac{\omega_{n-1}(\Theta)-n+1}{n} \tag{1.1}$$

hold for any point $\Theta = (\theta_1, \ldots, \theta_n)$ in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$.

Moreover, Jarník [10, 11] established that both inequalities in (1.1) are optimal, and, consequently, that Theorem K is best possible. It is the main purpose of the present paper to show that, however, Theorem K can be refined if we introduce two further quantities associated with Θ.

Following the general "hat" notations of [2], let us introduce the uniform analogues of the exponents $\omega_0(\Theta)$ and $\omega_{n-1}(\Theta)$.

2000 *Mathematics Subject Classification :* 11J13.

Definition 2. *We denote respectively by $\hat{\omega}_0(\Theta)$ and $\hat{\omega}_{n-1}(\Theta)$ the supremum of the real numbers ω such that for all sufficiently large real number X, there exists a non-zero integer $(n+1)$-tuples $(x_0, \ldots, x_n)$ with supremum norm*

$$\max_{0 \le i \le n} |x_i| \le X,$$

satisfying respectively the inequation

$$\max_{1 \le i \le n} |x_0\theta_i - x_i| \le X^{-\omega} \quad \textit{or} \quad |x_0 + x_1\theta_1 + \cdots + x_n\theta_n| \le X^{-\omega}.$$

We establish the following refinement of Khintchine's theorem, which involves the uniform exponents associated with Θ.

Theorem 1. *Suppose $n \ge 2$. The inequalities*

$$\frac{(\hat{\omega}_{n-1}(\Theta) - 1)\omega_{n-1}(\Theta)}{((n-2)\hat{\omega}_{n-1}(\Theta) + 1)\omega_{n-1}(\Theta) + (n-1)\hat{\omega}_{n-1}(\Theta)} \le \omega_0(\Theta)$$

and

$$\omega_0(\Theta) \le \frac{(1 - \hat{\omega}_0(\Theta))\omega_{n-1}(\Theta) - n + 2 - \hat{\omega}_0(\Theta)}{n-1}$$

hold for any point $\Theta = (\theta_1, \ldots, \theta_n)$ in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$.

The above inequalities are stronger than (1.1), since

$$\hat{\omega}_{n-1}(\Theta) \ge n \quad \text{and} \quad \hat{\omega}_0(\Theta) \ge \frac{1}{n},$$

by the Dirichlet Box Principle. Theorem 1 was first established when $n = 2$ in [13] and its statement was announced in [4] and in [14]. It follows from the description given in [13] of the set of all possible quadruples

$$\big(\omega_1(\Theta), \omega_0(\Theta), \hat{\omega}_1(\Theta), \hat{\omega}_0(\Theta)\big),$$

where Θ ranges over $\mathbf{R}^2$, that Theorem 1 is optimal in dimension two.

Theorem K was extended by Dyson [7] to transfer inequalities between approximation to a system of linear forms and approximation of the tranpose system. It would be interesting to establish a suitable extension of Theorem 1.

The present paper is organized as follows. In Section 2, we define further exponents $\omega_d(\Theta)$ for $d = 1, \ldots, n-2$, measuring the accuracy with which Θ can be approximated by rational linear subvarieties of dimension d. We state in Theorems 2 and 3 transference inequalities linking $\omega_d(\Theta)$ and $\omega_{d+1}(\Theta)$, the composition of which gives Theorem K. This was already known [17, 14], but our proof, based on the second theorem of Minkowski, is

new. Furthermore, our method allows us to refine inequalities between $\omega_0(\Theta)$ and $\omega_1(\Theta)$ (resp. between $\omega_{n-1}(\Theta)$ and $\omega_{n-2}(\Theta)$), by taking also $\hat{\omega}_0(\Theta)$ (resp. $\hat{\omega}_{n-1}(\Theta)$) into account. Using this, we get Theorem 1, as is explained in Section 7. Section 3 is devoted to some preliminaries of multilinear algebra. In Section 4 and at the beginning of Section 6, we give alternative definitions of the exponents ω_d. Theorems 2 and 3 are established in Sections 5 and 6, respectively.

2. Going-up and going-down transfers

It is convenient to view $\mathbf{R}^n$ as a subset of $\mathbf{P}^n(\mathbf{R})$ via the usual embedding $(x_1, \ldots, x_n) \mapsto (1, x_1, \ldots, x_n)$. We shall identify $\Theta = (\theta_1, \ldots, \theta_n)$ with its image in $\mathbf{P}^n(\mathbf{R})$.

Following [14], let us introduce for each integer d with $0 \le d \le n-1$ an exponent $\omega_d(\Theta)$ which measures the approximation to the point $\Theta \in \mathbf{P}^n(\mathbf{R})$ by rational linear projective subvarieties of dimension d, in terms of their height. Denote by d the projective distance on $\mathbf{P}^n(\mathbf{R})$ (it will be defined in §4 below ; notice however that the normalization used there does not matter for our purpose). For any real linear subvariety L of $\mathbf{P}^n(\mathbf{R})$, we denote by

$$\mathrm{d}(\Theta, L) = \min_{P \in L} \mathrm{d}(\Theta, P)$$

the minimal distance between Θ and the real points P of L. When L is rational over $\mathbf{Q}$, we indicate moreover by $H(L)$ its height, that is the Weil height of any system of Plücker coordinates of L. It is convenient to normalize the height by using the Euclidean norm at the Archimedean place of $\mathbf{Q}$. We refer to §1 of [17] for more information on the notion of height of a linear subspace.

Definition 3. *Let d be an integer with $0 \le d \le n-1$. We denote by $\omega_d(\Theta)$ the supremum of the real numbers ω for which there exist infinitely many rational linear subvarieties $L \subset \mathbf{P}^n(\mathbf{R})$ such that*

$$\dim(L) = d \quad \textit{and} \quad \mathrm{d}(\Theta, L) \le H(L)^{-1-\omega}.$$

Definitions 1 and 3 are consistent, since $\mathrm{d}(\Theta, L)$ compares respectively with

$$\max_{1 \le i \le n} \left| \theta_i - \frac{x_i}{x_0} \right| \quad \text{and} \quad \frac{|y_0 + y_1\theta_1 + \cdots + y_n\theta_n|}{\max_{0 \le i \le n} |y_i|}$$

when L is either the rational point (case $d = 0$) with homogeneous coordinates $(1, x_1/x_0, \ldots, x_n/x_0)$, or the hyperplane (when $d = n-1$) with homogeneous equation $y_0X_0 + \cdots + y_nX_n = 0$.

Theorem 1 is a consequence of the following two statements.

Theorem 2 (Going-up transfer). *Let $\Theta = (\theta_1, \dots, \theta_n)$ be in $\mathbf{R}^n$ with $1, \theta_1, \dots, \theta_n$ linearly independent over $\mathbf{Q}$. For any integer d with $0 \leq d \leq n-2$, we have the lower bound*

$$\omega_{d+1}(\Theta) \geq \frac{(n-d)\omega_d(\Theta) + 1}{n-d-1}. \tag{2.1}$$

Furthermore,

$$\omega_1(\Theta) \geq \frac{\omega_0(\Theta) + \hat{\omega}_0(\Theta)}{1 - \hat{\omega}_0(\Theta)}. \tag{2.2}$$

Theorem 3 (Going-down transfer). *Let $\Theta = (\theta_1, \dots, \theta_n)$ be in $\mathbf{R}^n$ with $1, \theta_1, \dots, \theta_n$ linearly independent over $\mathbf{Q}$. For any integer d with $1 \leq d \leq n-1$, we have the lower bound*

$$\omega_{d-1}(\Theta) \geq \frac{d\,\omega_d(\Theta)}{\omega_d(\Theta) + d + 1}. \tag{2.3}$$

Furthermore,

$$\omega_{n-2}(\Theta) \geq \frac{(\hat{\omega}_{n-1}(\Theta) - 1)\omega_{n-1}(\Theta)}{\omega_{n-1}(\Theta) + \hat{\omega}_{n-1}(\Theta)}. \tag{2.4}$$

The lower bounds (2.1) and (2.3) are implicit in [17] and are stated in [14]. It is shown in [14] that their composition produces Khintchine's theorem. The same splitting principle is used here. We prove Theorem 1 in §7 by iterating successively the finer Going-up estimates (2.2) and (2.1), and in the other direction the Going-down inequalities (2.4) and (2.3).

In contrast with the previous works [13, 14, 17], our approch is based here on the use of the second theorem of Minkowski on the successive minima of a convex body, combined with Mahler's theory of compound convex bodies [16].

We conclude this section by formulating the transfer inequalities between $\omega_d(\Theta)$ and $\omega_{d'}(\Theta)$ that easily follow from repeated applications of (2.1) and (2.3).

Corollary 1. *Let $\Theta = (\theta_1, \dots, \theta_n)$ be in $\mathbf{R}^n$ with $1, \theta_1, \dots, \theta_n$ linearly independent over $\mathbf{Q}$. For any integers d, d' with $0 \leq d < d' \leq n-1$, we have*

$$\frac{(d+1)\omega_{d'}(\Theta)}{(d'-d)\omega_{d'}(\Theta) + d' + 1} \leq \omega_d(\Theta) \leq \frac{(n-d')\omega_{d'}(\Theta) - d' + d}{n-d}.$$

3. Multilinear algebra

We collect in this section some classical results of multilinear algebra and their geometrical interpretation in terms of join and intersection of linear varieties in the space $\mathbf{R}^{n+1}$. For more details, we refer to [1].

First, we equip the real vector space $\mathbf{R}^{n+1}$ with the usual scalar product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_{n+1} y_{n+1}, \quad \mathbf{x} = (x_1, \ldots, x_{n+1}), \quad \mathbf{y} = (y_1, \ldots, y_{n+1}),$$

and extend it naturally to the Grassmann algebra $\Lambda(\mathbf{R}^{n+1})$, by requiring that for any orthonormal basis $\{\mathbf{e}_i\}_{1 \le i \le n+1}$ of $\mathbf{R}^{n+1}$, the family of wedge products

$$\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_r}\,; \quad 1 \le i_1 < \cdots < i_r \le n+1,\ 0 \le r \le n+1,$$

is an orthonormal basis of $\Lambda(\mathbf{R}^{n+1})$. Then, the Cauchy-Binet formula shows that

$$\mathbf{X} \cdot \mathbf{Y} = \det \Big(\mathbf{x}_i \cdot \mathbf{y}_j \Big)_{1 \le i,j \le r} \tag{3.1}$$

for any pair of decomposable r-vectors $\mathbf{X} = \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_r$ and $\mathbf{Y} = \mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_r$. The scalar product $\cdot$ enables us to identify the dual of the real vector space $\Lambda^r(\mathbf{R}^{n+1})$ with itself. For any multivector $\mathbf{X} \in \Lambda(\mathbf{R}^{n+1})$, we denote by $|\mathbf{X}| = \sqrt{\mathbf{X} \cdot \mathbf{X}}$ the Euclidean norm of $\mathbf{X}$.

Let $\mathbf{X} \in \Lambda^r(\mathbf{R}^{n+1})$ and $\mathbf{Y} \in \Lambda^s(\mathbf{R}^{n+1})$ be two multivectors of respective degree r and s with $s \le r$. We define the *internal product* (also called *contraction*) of $\mathbf{X}$ by $\mathbf{Y}$, as the unique multivector

$$\mathbf{Y} \lrcorner \mathbf{X} \in \Lambda^{r-s}(\mathbf{R}^{n+1})$$

for which the equality

$$\mathbf{Z} \cdot (\mathbf{Y} \lrcorner \mathbf{X}) = (\mathbf{Z} \wedge \mathbf{Y}) \cdot \mathbf{X} \tag{3.2}$$

holds for any $\mathbf{Z} \in \Lambda^{r-s}(\mathbf{R}^{n+1})$. In other words, the application $\mathbf{X} \mapsto \mathbf{Y} \lrcorner \mathbf{X}$ is the transpose of the linear map $\mathbf{Z} \mapsto \mathbf{Z} \wedge \mathbf{Y}$ with respect to the dot pairing.

Assume now that $\mathbf{X} = \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_r$ and $\mathbf{Y} = \mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_s$ are decomposable multivectors. When $s = 1$, we deduce from (3.1) and (3.2) the explicit formula

$$\mathbf{y} \lrcorner \mathbf{X} = \sum_{j=1}^{r} (-1)^{r-j} (\mathbf{y} \cdot \mathbf{x}_j) \mathbf{x}_1 \wedge \ldots \wedge \hat{\mathbf{x}}_j \wedge \ldots \wedge \mathbf{x}_r \tag{3.3}$$

for any vector $\mathbf{y} \in \Lambda^1(\mathbf{R}^{n+1})$. It formally follows from (3.2) that

$$(\mathbf{Y} \wedge \mathbf{Y}') \lrcorner \mathbf{X} = \mathbf{Y} \lrcorner (\mathbf{Y}' \lrcorner \mathbf{X}) \tag{3.4}$$

for any pair of multivectors $\mathbf{Y}$ and $\mathbf{Y}'$ with respective degree s and s' such that $s+s' \leq r$. Starting with (3.3) and using (3.4), we obtain by induction on s the formula

$$\mathbf{Y} \lrcorner \mathbf{X} = \sum \operatorname{sgn}(\sigma)(\mathbf{y}_1 \cdot \mathbf{x}_{\sigma(r-s+1)}) \cdots (\mathbf{y}_s \cdot \mathbf{x}_{\sigma(r)})\mathbf{x}_{\sigma(1)} \wedge \ldots \wedge \mathbf{x}_{\sigma(r-s)} \tag{3.5}$$

where the sum is taken over all the substitutions σ of $\{1, \ldots, r\}$ such that $\sigma(1) < \cdots < \sigma(r-s)$.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ be any positively oriented (meaning that $\det(\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}) = 1$) orthonormal basis of $\mathbf{R}^{n+1}$. Remark that the volume form $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1}$ does not depend upon the choice of such a basis.

Definition 4. *For every $\mathbf{X}$ in $\Lambda^r(\mathbf{R}^{n+1})$, we denote by*

$$*\mathbf{X} = \mathbf{X} \lrcorner (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1}) \in \Lambda^{n+1-r}(\mathbf{R}^{n+1})$$

the Hodge dual of $\mathbf{X}$.

Expanding

$$\mathbf{X} = \sum_{1 \leq i_1 < \cdots < i_r \leq n+1} X_{i_1, \ldots, i_r} \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_r}$$

in the induced orthonormal basis of $\Lambda^r(\mathbf{R}^{n+1})$, we find

$$*\mathbf{X} = \sum_{1 \leq i_1 < \cdots < i_r \leq n+1} \varepsilon_{i_1, \ldots, i_r} X_{i_1, \ldots, i_r} \mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{n+1-r}},$$

where $\{j_1, \ldots, j_{n+1-r}\} = \{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_r\}$ with $j_1 < \cdots < j_{n+1-r}$, and $\varepsilon_{i_1, \ldots, i_r}$ stands for the signature of the shuffle substitution $(1, \ldots, n+1) \mapsto (j_1, \ldots, j_{n+1-r}, i_1, \ldots, i_r)$. The Hodge star operator

$$* : \Lambda^r(\mathbf{R}^{n+1}) \xrightarrow{\sim} \Lambda^{n+1-r}(\mathbf{R}^{n+1})$$

is clearly an isometry for the dot scalar product and iterating twice the Hodge star, we get

$$* \circ * = (-1)^{r(n+1-r)} \mathrm{Id}. \tag{3.6}$$

Lemma 1. *Let $\mathbf{X} = \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_r$ be a system of Plücker coordinates* ($\natural$) *of a r-dimensional subspace*

$$V = < \mathbf{x}_1, \ldots, \mathbf{x}_r >$$

($\natural$) The word “coordinates” classically refers to the canonical basis of $\Lambda^r(\mathbf{R}^{n+1})$.

in $\mathbf{R}^{n+1}$. Then $*\mathbf{X}$ is a system of Plücker coordinates of the orthogonal $V^\perp$ of V.

Proof. That is the assertion of Theorem I of Chapter VII §3 in [9]. Using the notion of contraction, we may argue as follows. Take any orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ of V and extend it to an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ of $\mathbf{R}^{n+1}$. Then

$$\mathbf{X} = \rho(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_r)$$

for some non-zero real number ρ. Now, it follows from (3.5) that

$$*\mathbf{X} = \pm\rho(\mathbf{e}_{r+1} \wedge \ldots \wedge \mathbf{e}_{n+1}).$$

□

Remark. The same argument shows more generally that if $\mathbf{Y} = \mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_s$ is a system of Plücker coordinates of an s-dimensional vector space $W =< \mathbf{y}_1, \ldots, \mathbf{y}_s >$ with $s \geq r$, then $\mathbf{X} \lrcorner \mathbf{Y}$ is a system of Plücker coordinates of the intersection $W \cap V^\perp$, provided that this intersection has dimension $s - r$.

Lemma 2. *For any $\mathbf{X} \in \Lambda^r(\mathbf{R}^{n+1})$ and $\mathbf{Y} \in \Lambda^s(\mathbf{R}^{n+1})$ with $r + s \leq n + 1$, we have the duality formulæ*

$$*(\mathbf{Y} \wedge \mathbf{X}) = \mathbf{Y} \lrcorner (*\mathbf{X})$$

Proof. Using (3.4), we find

$$*(\mathbf{Y} \wedge \mathbf{X}) = (\mathbf{Y} \wedge \mathbf{X}) \lrcorner (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1}) = \mathbf{Y} \lrcorner (\mathbf{X} \lrcorner (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1})) = \mathbf{Y} \lrcorner (*\mathbf{X}).$$

□

4. Alternative definition of the intermediate exponents

Let P and Q be points in $\mathbf{P}^n(\mathbf{R})$ with homogeneous coordinates $\mathbf{x}$ and $\mathbf{y}$. As in [14], we define the projective distance $\mathrm{d}(P, Q)$ between P and Q by

$$\mathrm{d}(P, Q) = \frac{|\mathbf{x} \wedge \mathbf{y}|}{|\mathbf{x}||\mathbf{y}|}.$$

It has been shown in Lemma 1 of [14] that for any point Θ in $\mathbf{P}^n(\mathbf{R})$ with homogeneous coordinates $\mathbf{y} = (1, \theta_1, \ldots, \theta_n)$ and any linear subvariety L of $\mathbf{P}^n(\mathbf{R})$ with Plücker coordinates $\mathbf{X}$, the minimal distance $\mathrm{d}(\Theta, L)$ between Θ and the set of real points of L is equal to

$$\mathrm{d}(\Theta, L) = \frac{|\mathbf{y} \wedge \mathbf{X}|}{|\mathbf{y}||\mathbf{X}|}. \tag{4.1}$$

We can now reformulate Definition 3 in terms of integer solutions of the following system of linear inequations.

Proposition. *For any integer d with $0 \le d \le n-1$, the exponent $\omega_d(\Theta)$ is the supremum of the real numbers ω for which there exist infinitely many integer multivectors $\mathbf{X} \in \Lambda^{d+1}(\mathbf{Z}^{n+1})$ such that*

$$|\mathbf{y} \wedge \mathbf{X}| \le |\mathbf{X}|^{-\omega}.$$

In relation with Definition 4 of [14], we do not assume here that the multivectors $\mathbf{X}$ occurring in the Proposition are decomposable. To suppress this additional condition, we expand the remark given on page 312 of [14]. The following lemma will be as well our main ingredient to prove Theorem 2.

Lemma 3. *Let $\mathbf{y} = (1, \theta_1, \ldots, \theta_n) \in \mathbf{R}^{n+1}$ and let U, V be positive real numbers with $V \le U$. The convex body $\mathcal{C}$ of $\Lambda^{d+1}(\mathbf{R}^{n+1})$ consisting of the $\mathbf{Z}$ such that*

$$|\mathbf{Z}| \le UV^d \quad \textit{and} \quad |\mathbf{y} \wedge \mathbf{Z}| \le V^{d+1} \tag{4.2}$$

is comparable (†) to the $(d+1)$-th compound of the convex body $\mathcal{C}'$ consisting of the $\mathbf{z} \in \mathbf{R}^{n+1}$ such that

$$|\mathbf{z}| \le U \quad \textit{and} \quad |\mathbf{y} \wedge \mathbf{z}| \le V. \tag{4.3}$$

Proof. The convex body $\mathcal{C}'$ is comparable to the parallelepiped $\mathcal{P}$ defined by

$$|x_0| \le U, \quad |x_0\theta_i - x_i| \le V, \quad 1 \le i \le n.$$

However, $\mathcal{P}$ is the convex hull of the points

$$\pm U\mathbf{y}, \pm V\mathbf{e}_1, \ldots, \pm V\mathbf{e}_n,$$

where

$$\mathbf{e}_1 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, \ldots, 0, 1).$$

The convex compound $\mathcal{C}'^{d+1}$ is then comparable to the convex hull in $\Lambda^{d+1}(\mathbf{R}^{n+1})$ of the exterior products of $d+1$ of these points, that is, of

$$\pm V^{d+1}\mathbf{e}_{i_0} \wedge \ldots \wedge \mathbf{e}_{i_d}, \quad 1 \le i_0 < \cdots < i_d \le n,$$

(†) We say that two families $\mathcal{C}_1$ and $\mathcal{C}_2$ of symmetrical convex bodies, parametrized by (say) U and V, are *comparable* if there exists a real number $\kappa > 1$, such that the inclusions $\kappa^{-1}\mathcal{C}_1(U,V) \subseteq \mathcal{C}_2(U,V) \subseteq \kappa\mathcal{C}_1(U,V)$ hold for any parameters U, V. Accordingly, the constants implied in the forthcoming symbols $\ll$, $\gg$ and $\asymp$ may depend on n and Θ, but not on U and V. The relation $f \asymp g$ means that we have both $f \ll g$ and $f \gg g$.

and

$$\pm UV^d \mathbf{y}\wedge \mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_d},\quad 1\le i_1<\cdots<i_d\le n.$$

The points $\mathbf{Z}$ of this form satisfy

$$|\mathbf{Z}|\ll UV^d,\quad |\mathbf{y}\wedge\mathbf{Z}|\ll V^{d+1}.$$

Conversely, let $\mathbf{Z}$ be in $\Lambda^{d+1}(\mathbf{R}^{n+1})$ for which (4.2) holds and express it in the base composed of the $d+1$ exterior products of the base $(\mathbf{y},\mathbf{e}_1,\ldots,\mathbf{e}_n)$, that is,

$$\mathbf{Z}=\sum a_{i_0,i_1,\ldots,i_d}\,\mathbf{e}_{i_0}\wedge\ldots\wedge\mathbf{e}_{i_d}+\sum b_{i_1,i_2,\ldots,i_d}\,\mathbf{y}\wedge\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_d}.$$

Then, we have the estimates

$$\sum|a_{i_0,i_1,\ldots,i_d}|+\sum|b_{i_1,i_2,\ldots,i_d}|\asymp|\mathbf{Z}|\le UV^d\quad\text{and}\quad\sum|a_{i_0,i_1,\ldots,i_d}|\asymp|\mathbf{y}\wedge\mathbf{Z}|\le V^{d+1}.$$

This completes the proof of the lemma. □

With this lemma, we are able to establish our Proposition.

Proof of the Proposition. Let ω be a real number with $\omega\ge -1$ and let $\mathbf{X}$ be a non-zero point in $\Lambda^{d+1}(\mathbf{Z}^{n+1})$ such that

$$|\mathbf{y}\wedge\mathbf{X}|\le|\mathbf{X}|^{-\omega}.$$

The first minimum of the convex body $\mathcal{C}$ composed of the $\mathbf{Z}\in\Lambda^{d+1}(\mathbf{R}^{n+1})$ such that

$$|\mathbf{Z}|\le|\mathbf{X}|\quad\text{and}\quad|\mathbf{y}\wedge\mathbf{Z}|\le|\mathbf{X}|^{-\omega}$$

is therefore at most equal to 1 since $\mathbf{X}$ belongs to $\mathcal{C}$. Setting

$$U=|\mathbf{X}|^{(d\omega+d+1)/(d+1)},\quad V=|\mathbf{X}|^{-\omega/(d+1)},\tag{4.4}$$

we observe that $V\le U$ and that

$$|\mathbf{X}|=UV^d,\quad|\mathbf{X}|^{-\omega}=V^{d+1}.$$

By Lemma 3, the convex $\mathcal{C}$ is comparable to the $(d+1)$-th compound of the convex body $\mathcal{C}'\subset\mathbf{R}^{n+1}$ defined by the inequalities (4.3). Now, Mahler's theory on compound convex bodies tells us that the integer point where $\mathcal{C}$ reaches its first minimum is essentially obtained as the wedge product $\mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_{d+1}$ of the integer points $\mathbf{x}_i, 1\le i\le d+1$, where $\mathcal{C}'$ reaches its i-th minimum. We may therefore assume that $\mathbf{X}=\mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_{d+1}$.

Let $L \subset \mathbf{P}^n(\mathbf{R})$ be the d-dimensional rational linear subvariety $L = \mathbf{P}(V)$ where $V = \langle \mathbf{x}_1, \ldots, \mathbf{x}_{d+1} \rangle$. By (4.1), we obtain

$$\mathrm{d}(\Theta, L) = \frac{|\mathbf{y} \wedge \mathbf{X}|}{|\mathbf{y}||\mathbf{X}|} \leq |\mathbf{y}|^{-1} |\mathbf{X}|^{-1-\omega} \ll H(L)^{-1-\omega},$$

so that $\omega_d(\Theta) \geq \omega$.

Conversely, if L satisfies $\mathrm{d}(\Theta, L) \leq H(L)^{-1-\omega}$, choose a system of coprime integer Plücker coordinates $\mathbf{X}$ of L, so that $H(L) = |\mathbf{X}|$. Then (4.1) shows that the upper bound $|\mathbf{y} \wedge \mathbf{X}| \ll |\mathbf{X}|^{-\omega}$ holds true. □

Our Proposition enables us to recover the following corollary, which was already obtained in [14] and earlier in [17], using different arguments.

Corollary 2. *For any integer d with $0 \leq d \leq n-1$, we have the lower bound*

$$\omega_d(\Theta) \geq \frac{d+1}{n-d}.$$

Proof. The map $\Lambda^{d+1}(\mathbf{R}^{n+1}) \longrightarrow \Lambda^{d+2}(\mathbf{R}^{n+1})$ which sends $\mathbf{X} \mapsto \mathbf{y} \wedge \mathbf{X}$ has rank $\binom{n+1}{d+1} - \binom{n}{d}$. Applying the Box Principle to the system of linear forms $\mathbf{y} \wedge \mathbf{X}$ in the integer variables $\mathbf{X}$, we find that

$$\omega_d(\Theta) \geq \frac{\binom{n+1}{d+1} - \left(\binom{n+1}{d+1} - \binom{n}{d}\right)}{\binom{n+1}{d+1} - \binom{n}{d}} = \frac{\binom{n}{d}}{\binom{n+1}{d+1} - \binom{n}{d}} = \frac{d+1}{n-d},$$

as claimed. □

5. Proof of Theorem 2

We use the Proposition as a more convenient characterization of the exponents $\omega_d(\Theta)$ and take again the notations of Section 4. Let ω be a real number with $-1 \leq \omega < \omega_d(\Theta)$ and let $\mathbf{X} \in \Lambda^{d+1}(\mathbf{Z}^{n+1})$ be such that

$$|\mathbf{y} \wedge \mathbf{X}| \leq |\mathbf{X}|^{-\omega},$$

where $\mathbf{y}$ denotes the homogeneous coordinates of Θ. Recall that U and V are given by (4.4) and that the convex bodies $\mathcal{C}$ and $\mathcal{C}'$ are defined by (4.2) and (4.3), respectively. The first minimum λ_1 of the convex body $\mathcal{C}$ is at most equal to 1 since $\mathbf{X}$ belongs to $\mathcal{C}$. Replacing possibly $\mathbf{X}$ by the integer point where this first minimum is reached and increasing suitably ω, we may assume that $\lambda_1 = 1$.

By Lemma 3, the convex $\mathcal{C}$ is comparable to the $(d+1)$-th compound of the convex body $\mathcal{C}'$ of volume

$$\mathrm{vol}(\mathcal{C}') \asymp UV^n = |\mathbf{X}|^{(-(n-d)\omega+d+1)/(d+1)}.$$

By Minkowski's Theorem, the successive minima $\lambda'_1 \leq \ldots \leq \lambda'_{n+1}$ of $\mathcal{C}'$ satisfy

$$\lambda'_1 \times \ldots \times \lambda'_{n+1} \asymp \mathrm{vol}(\mathcal{C}')^{-1} \asymp |\mathbf{X}|^{((n-d)\omega-d-1)/(d+1)}.$$

Since $\mathcal{C}$ is comparable to the $(d+1)$-th compound of $\mathcal{C}'$, Mahler's theorem on compound convex bodies asserts that λ_1, the first minimum of $\mathcal{C}$, is comparable to the product $\lambda'_1 \times \ldots \times \lambda'_{d+1}$. Consequently,

$$\lambda'_1 \times \ldots \times \lambda'_{d+1} \asymp 1 \tag{5.1}$$

and

$$(\lambda'_{d+2})^{n-d} \leq \lambda'_{d+2} \times \ldots \times \lambda'_{n+1} \asymp |\mathbf{X}|^{((n-d)\omega-d-1)/(d+1)},$$

whence

$$\lambda'_{d+2} \ll |\mathbf{X}|^{((n-d)\omega-d-1)/((d+1)(n-d))}. \tag{5.2}$$

Now, since the $(d+2)$-th compound of $\mathcal{C}'$ has its first minimum comparable to

$$\lambda'_1 \times \ldots \times \lambda'_{d+2} \asymp \lambda'_{d+2},$$

it follows from Lemma 3 that there exists $\mathbf{X}' \in \Lambda^{d+2}(\mathbf{Z}^{n+1})$ such that

$$|\mathbf{X}'| \ll \lambda'_{d+2} UV^{d+1}, \quad |\mathbf{y} \wedge \mathbf{X}'| \ll \lambda'_{d+2} V^{d+2}.$$

A rapid computation using (5.2) yields that

$$\lambda'_{d+2} UV^{d+1} \ll |\mathbf{X}|^{(n-d-1)/(n-d)}$$

and

$$\lambda'_{d+2} V^{d+2} \ll |\mathbf{X}|^{-((n-d)\omega+1)/(n-d)}.$$

This gives

$$|\mathbf{y} \wedge \mathbf{X}'| \ll |\mathbf{X}'|^{-((n-d)\omega+1)/(n-d-1)},$$

and we get (2.1) since ω can be taken arbitrarily close to $\omega_d(\Theta)$.

To establish (2.2), let us first observe that (5.2) with $d=0$ gives

$$\lambda'_2 \ll |\mathbf{X}|^{\omega-1/n}. \tag{5.3}$$

One can get a better upper bound for λ'_2 when $d=0$ by taking the uniform exponents into account, as we show now. In that case $\mathcal{C}=\mathcal{C}'$ and $\lambda'_1=\lambda_1=1$. The vector $\mathbf{X}$ is necessarily primitive in $\mathbf{Z}^{n+1}$, since the convex body $\mathcal{C}'$ attains its first minimum at that point. Let $\hat{\omega}$ be a real number with $\hat{\omega}<\hat{\omega}_0(\Theta)$. By Definition 2, there exists a non-zero integer point $\mathbf{x}$ such that

$$|\mathbf{x}|<|\mathbf{X}|, \quad |\mathbf{y}\wedge\mathbf{x}|\le|\mathbf{X}|^{-\hat{\omega}}.$$

Since $\mathbf{X}$ is primitive, the vectors $\mathbf{x}$ and $\mathbf{X}$ are linearly independent. This gives

$$\lambda'_2\ll|\mathbf{X}|^{\omega-\hat{\omega}}. \tag{5.4}$$

Note that the upper estimate (5.4) may be sharper than (5.3) since $\hat{\omega}_0(\Theta)\ge 1/n$.

Observing that $U=|\mathbf{X}|$ and $V=|\mathbf{X}|^{-\omega}$ and proceeding as above, we infer from (5.4) that

$$\lambda'_2UV\ll|\mathbf{X}|^{1-\hat{\omega}}$$

and

$$\lambda'_2V^2\ll|\mathbf{X}|^{-(\omega+\hat{\omega})},$$

whence

$$|\mathbf{y}\wedge\mathbf{X}'|\ll|\mathbf{X}'|^{-(\omega+\hat{\omega})/(1-\hat{\omega})}.$$

Letting ω tends to $\omega_0(\Theta)$ and $\hat{\omega}$ tends to $\hat{\omega}_0(\Theta)$, this gives

$$\omega_1(\Theta)\ge\frac{\omega_0(\Theta)+\hat{\omega}_0(\Theta)}{1-\hat{\omega}_0(\Theta)}.$$

We have proved (2.2).

6. Proof of Theorem 3

The proof is parallel to that of Theorem 2. We use Hodge duality to reverse the Going-down transfer into a Going-up transfer, noting that the duality permutes the dimension with the codimension.

Let us start with the following dual version of the above Proposition.

Lemma 4. *For $d=0,\dots,n-1$, the exponent $\omega_d(\Theta)$ of a point Θ in $\mathbf{R}^n$ with homogeneous coordinates $\mathbf{y}$ is the supremum of the real numbers ω such that there are infinitely many $\mathbf{X}\in\Lambda^{n-d}(\mathbf{Z}^{n+1})$ with*

$$|\mathbf{y}\lrcorner\mathbf{X}|\le|\mathbf{X}|^{-\omega}.$$

Proof. By Lemma 2 and (3.6), we have

$$*(\mathbf{y}\wedge *\mathbf{X})=(-1)^{(d+1)(n-d)}(\mathbf{y}\lrcorner\mathbf{X}),$$

for every $\mathbf{X}$ in $\Lambda^{n-d}(\mathbf{R}^{n+1})$. Note that $*$ maps $\Lambda^{n-d}(\mathbf{Z}^{n+1})$ isometrically onto $\Lambda^{d+1}(\mathbf{Z}^{n+1})$, so that

$$|*\mathbf{X}| = |\mathbf{X}| \quad \text{and} \quad |\mathbf{y} \wedge *\mathbf{X}| = |\mathbf{y} \lrcorner \mathbf{X}|.$$

Now, replace $\mathbf{X}$ by $*\mathbf{X}$ in the Proposition to conclude. $\square$

Here is now the dual version of Lemma 3.

Lemma 5. *Let d be an integer with $0 \le d \le n-1$ and let U, V be positive real numbers with $V \le U$. The convex body $\mathcal{C}$ of $\Lambda^{n-d}(\mathbf{R}^{n+1})$ consisting of the $\mathbf{Z}$ such that*

$$|\mathbf{Z}| \le U^{n-d} \quad \text{and} \quad |\mathbf{y} \lrcorner \mathbf{Z}| \le U^{n-d-1}V \tag{6.1}$$

is comparable to the $(n-d)$-th compound of the convex body $\mathcal{C}'$ composed of the $\mathbf{z} \in \mathbf{R}^{n+1}$ such that

$$|\mathbf{z}| \le U \quad \text{and} \quad |\mathbf{y} \cdot \mathbf{z}| \le V. \tag{6.2}$$

Proof. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of the orthogonal of $\mathbf{y}$ in $\mathbf{R}^{n+1}$. The convex body $\mathcal{C}'$ is comparable to the parallelepiped $\mathcal{P}$ consisting of the points

$$x_0\mathbf{y} + x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \quad \text{where} \quad |x_0| \le V, \quad |x_i| \le U, \quad 1 \le i \le n.$$

Note that $\mathcal{P}$ is comparable to the convex hull of the points

$$\pm V\mathbf{y}, \pm U\mathbf{e}_1, \dots, \pm U\mathbf{e}_n.$$

The compound convex body $\mathcal{C}'^{n-d}$ is then comparable to the convex hull in $\Lambda^{n-d}(\mathbf{R}^{n+1})$ of the exterior products of $n-d$ of these points, that is, of

$$\pm U^{n-d}\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d}}, \quad 1 \le i_1 < \cdots < i_{n-d} \le n, \tag{6.3}$$

and

$$\pm U^{n-d-1}V\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d-1}} \wedge \mathbf{y}, \quad 1 \le i_1 < \cdots < i_{n-d-1} \le n. \tag{6.4}$$

Express now any point $\mathbf{Z}$ in $\Lambda^{n-d}(\mathbf{R}^{n+1})$ in the base composed of the $n-d$ exterior products of the base $(\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{y})$, that is,

$$\mathbf{Z} = \sum a_{i_1,\dots,i_{n-d}}\, \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d}} + \sum b_{i_1,\dots,i_{n-d-1}}\, \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d-1}} \wedge \mathbf{y}.$$

Then, formula (3.3) shows that

$$\mathbf{y} \lrcorner \mathbf{Z} = |\mathbf{y}|^2 \left(\sum b_{i_1,\dots,i_{n-d-1}}\, \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d-1}} \right).$$

Therefore, the points $\mathbf{Z}$ of the form (6.3) or (6.4) satisfy

$$(6.5) \qquad |\mathbf{Z}| \le U^{n-d} \quad \text{and} \quad |\mathbf{y} \lrcorner \mathbf{Z}| \le |\mathbf{y}|^2 U^{n-d-1} V.$$

Conversely, for any point $\mathbf{Z}$ satisfying (6.5), the coefficients $a_{i_1,\dots,i_{n-d}}$ (resp. $b_{i_1,\dots,i_{n-d-1}}$) are bounded in absolute value by U^{n-d} (resp. by $U^{n-d-1}V$). This completes the proof of the lemma. □

With these two lemmata, we are able to establish Theorem 3.

Proof of Theorem 3. Let ω be a real number with $-1 \le \omega < \omega_d(\Theta)$. By Lemma 4, there exist infinitely many points $\mathbf{X} \in \Lambda^{n-d}(\mathbf{Z}^{n+1})$ such that

$$|\mathbf{y} \lrcorner \mathbf{X}| \le |\mathbf{X}|^{-\omega}.$$

Fix such a point $\mathbf{X}$ with large norm $|\mathbf{X}|$ and consider the convex body $\mathcal{C}$ composed of the multivectors $\mathbf{Z} \in \Lambda^{n-d}(\mathbf{R}^{n+1})$ such that

$$|\mathbf{Z}| \le |\mathbf{X}| \quad \text{and} \quad |\mathbf{y} \lrcorner \mathbf{Z}| \le |\mathbf{X}|^{-\omega}.$$

It contains the integer point $\mathbf{X}$. Replacing possibly $\mathbf{X}$ by a smaller point and enlarging suitably ω, one can assume that $\mathbf{X}$ is the smallest non-zero integer point in $\mathcal{C}$. Thus, we may assume that the first minimum of $\mathcal{C}$ is equal to 1. Setting

$$U = |\mathbf{X}|^{1/(n-d)} \quad \text{and} \quad V = |\mathbf{X}|^{-((n-d)\omega+n-d-1)/(n-d)},$$

we observe that $V \le U$ and that

$$|\mathbf{X}| = U^{n-d}, \quad |\mathbf{X}|^{-\omega} = U^{n-d-1}V.$$

By Lemma 5, the convex body $\mathcal{C}$ is therefore comparable to the $(n-d)$-th compound of the convex body $\mathcal{C}'$ consisting of the real $(n+1)$-tuples $\mathbf{z}$ such that

$$|\mathbf{z}| \le U \quad \text{and} \quad |\mathbf{y} \cdot \mathbf{z}| \le V.$$

Let

$$\lambda_1 \le \ldots \le \lambda_{n+1}$$

be the successive minima of the convex body $\mathcal{C}'$. Since the Euclidean volume of $\mathcal{C}'$ is $\asymp U^n V$, the second theorem of Minkowski gives

$$\lambda_1 \times \ldots \times \lambda_{n+1} \asymp (U^n V)^{-1} = |\mathbf{X}|^{((n-d)\omega-d-1)/(n-d)}.$$

Since the first minimum of the $(n-d)$-th compound of $\mathcal{C}'$ is comparable to 1, one gets

$$\lambda_1 \times \ldots \times \lambda_{n-d} \asymp 1,$$

hence

$$\lambda_{n-d+1} \times \ldots \times \lambda_{n+1} \asymp |\mathbf{X}|^{((n-d)\omega-d-1)/(n-d)}.$$

Consequently,

$$\lambda_{n-d+1}^{d+1} \ll |\mathbf{X}|^{((n-d)\omega-d-1)/(n-d)}, \tag{6.6}$$

and

$$\lambda_{n-d+1} U \ll |\mathbf{X}|^{\omega/(d+1)}.$$

Since the first minimum of the $(n-d+1)$-th compound of $\mathcal{C}'$ is comparable to the product $\lambda_1 \times \ldots \times \lambda_{n-d+1}$, hence to λ_{n-d+1}, we infer from Lemma 5 that there exists a non-zero integer point $\mathbf{X}' \in \Lambda^{n-d+1}(\mathbf{Z}^{n+1})$ such that

$$|\mathbf{X}'| \ll \lambda_{n-d+1} U^{n-d+1} = \lambda_{n-d+1} U |\mathbf{X}| \ll |\mathbf{X}|^{(\omega+d+1)/(d+1)}$$

and

$$|\mathbf{y} \lrcorner \mathbf{X}'| \ll \lambda_{n-d+1} U^{n-d} V = \lambda_{n-d+1} U |\mathbf{X}|^{-\omega} \ll |\mathbf{X}|^{-d\omega/(d+1)}.$$

Since ω can be taken arbitrarily close to $\omega_d(\Theta)$, Lemma 4 gives (2.3).

For $d = n-1$, it is possible to get a sharper result. In that case $\mathcal{C} = \mathcal{C}'$ is a convex body in $\mathbf{R}^{n+1}$ and (6.6) reads

$$\lambda_2 \ll |\mathbf{X}|^{-1+\omega/n}. \tag{6.7}$$

Enlarging possibly ω, we may assume that

$$|\mathbf{y} \cdot \mathbf{X}| = |\mathbf{X}|^{-\omega}.$$

The vector $\mathbf{X}$ is necessarily primitive in $\mathbf{Z}^{n+1}$, since the convex body $\mathcal{C}'$ attains its first minimum at that point. Let $\hat{\omega}$ be a real number with $\hat{\omega} < \hat{\omega}_{n-1}(\Theta)$. By Definition 2, there exists a non-zero integer point $\mathbf{x} \in \mathbf{Z}^{n+1}$ such that

$$|\mathbf{x}| \le |\mathbf{X}|^{\omega/\hat{\omega}} \quad \text{and} \quad |\mathbf{y} \cdot \mathbf{x}| < |\mathbf{X}|^{-\omega}.$$

Since $\mathbf{X}$ is primitive, the vectors $\mathbf{x}$ and $\mathbf{X}$ are linearly independent ; otherwise $\mathbf{x}$ should be an integer multiple of $\mathbf{X}$ and $|\mathbf{y} \cdot \mathbf{x}|$ should be greater than or equal to $|\mathbf{y} \cdot \mathbf{X}| = |\mathbf{X}|^{-\omega}$. Thus, we obtain the upper bound

$$\lambda_2 \ll |\mathbf{X}|^{-1+\omega/\hat{\omega}}, \tag{6.8}$$

which may be better than (6.7) since $\hat{\omega}_{n-1}(\Theta) \geq n$. Now, we take again the preceding arguments. Noting that $U = |\mathbf{X}|$ and $V = |\mathbf{X}|^{-\omega}$, we obtain a non-zero point $\mathbf{X}' \in \Lambda^2(\mathbf{Z}^{n+1})$ satisfying

$$|\mathbf{X}'| \ll \lambda_2 U^2 \ll |\mathbf{X}|^{1+\omega/\hat{\omega}}$$

and

$$|\mathbf{y} \lrcorner \mathbf{X}'| \ll \lambda_2 UV \ll |\mathbf{X}|^{-\omega+\omega/\hat{\omega}}.$$

Then, Lemma 4 gives

$$\omega_{n-2}(\Theta) \geq \frac{(\hat{\omega}-1)\omega}{\omega+\hat{\omega}}.$$

Letting ω and $\hat{\omega}$ tend respectively to $\omega_{n-1}(\Theta)$ and $\hat{\omega}_{n-1}(\Theta)$, we have established (2.4).

7. Proof of Theorem 1

It is a formal consequence of the finer estimates (2.1)–(2.4).

Using the second inequality of Corollary 1 with $d = 1$ and $d' = n-1$, we get the estimate

$$\omega_{n-1}(\Theta) \geq (n-1)\omega_1(\Theta) + n - 2,$$

which, combined with (2.2), yields the right hand side of the claimed estimate, namely

$$\omega_{n-1}(\Theta) \geq (n-1)\frac{\omega_0(\Theta)+\hat{\omega}_0(\Theta)}{1-\hat{\omega}_0(\Theta)} + n - 2 = \frac{(n-1)\omega_0(\Theta)+\hat{\omega}_0(\Theta)+n-2}{1-\hat{\omega}_0(\Theta)}.$$

Using now the first inequality of Corollary 1 with $d = 0$ and $d' = n-2$, we get

$$\omega_0(\Theta) \geq \frac{\omega_{n-2}(\Theta)}{(n-2)\omega_{n-2}(\Theta)+n-1}$$

which, combined with (2.4), yields the claimed Going-down transfer inequality, namely

$$\omega_0(\Theta) \geq \frac{(\hat{\omega}_{n-1}(\Theta)-1)\omega_{n-1}(\Theta)}{((n-2)\hat{\omega}_{n-1}(\Theta)+1)\omega_{n-1}(\Theta)+(n-1)\hat{\omega}_{n-1}(\Theta)}.$$

References

[1] N. Bourbaki, Algebra 1, Chapter 3, Springer-Verlag, New-York, 1989.

[2] Y. Bugeaud and M. Laurent, *Exponents of Diophantine approximation and Sturmian continued fractions*, Ann. Inst. Fourier (Grenoble) 55 (2005), 773–804.

[3] Y. Bugeaud and M. Laurent, *Exponents of inhomogeneous Diophantine approximation*, Moscow Math. J. 5 (2005), 747–766.

[4] Y. Bugeaud and M. Laurent, *Exponents of Diophantine approximation.* In: Diophantine Geometry Proceedings, Scuola Normale Superiore Pisa, Ser. CRM, vol. 4, 2007, 101–121.

[5] Y. Bugeaud, Approximation by algebraic numbers. Cambridge Tracts in Mathematics, Cambridge, 2004.

[6] J. W. S. Cassels, An introduction to Diophantine Approximation. Cambridge Tracts in Math. and Math. Phys., vol. 99, Cambridge University Press, 1957.

[7] F. J. Dyson, *On simultaneous Diophantine approximations*, Proc. London Math. Soc. 49 (1947), 409–420.

[8] P. M. Gruber and C. G. Lekkerkerker, Geometry of numbers. Series Bibliotheca Mathematica 8, North–Holland, Amsterdam, 1987.

[9] H. Hodge and D. Pedoe, Methods of algebraic Geometry, Cambridge University Press, 1947.

[10] V. Jarník, *Über einen Satz von A. Khintchine*, Práce Mat.-Fiz. 43 (1935), 1–16.

[11] V. Jarník, *Über einen Satz von A. Khintchine, 2. Mitteilung*, Acta Arith. 2 (1936), 1–22.

[12] A. Ya. Khintchine, *Über eine Klasse linearer diophantischer Approximationen*, Rendiconti Circ. Mat. Palermo 50 (1926), 170–195.

[13] M. Laurent, *Exponents of Diophantine Approximation in dimension two*, Canad. J. Math. To appear in 2008.

[14] M. Laurent, *On transfer inequalities in Diophantine Approximation.* In: "Analytic Number Theory in Honour of Klaus Roth". Cambridge University Press (to appear in 2009), 306–314.

[15] K. Mahler, *Neuer Beweis einer Satz von A. Khintchine*, Mat. Sbornik 43 (1936), 961–962.

[16] K. Mahler, *On compound convex bodies, I*, Proc. London Math. Soc. (3) 5 (1955), 358–379.

[17] W. M. Schmidt, *On heights of algebraic subspaces and diophantine approximations*, Annals of Math. 85 (1967), 430–472.

[18] W. M. Schmidt, Diophantine Approximation. Lecture Notes in Math. 785, Springer, Berlin, 1980.

Yann Bugeaud
Université Louis Pasteur
U. F. R. de mathématiques
7, rue René Descartes
67084 STRASBOURG (FRANCE)

bugeaud@math.u-strasbg.fr

Michel Laurent
Institut de Mathématiques de Luminy
C.N.R.S. - U.M.R. 6206 - case 907
163, avenue de Luminy
13288 MARSEILLE CEDEX 9 (FRANCE)

laurent@iml.univ-mrs.fr